UNIT -II (DIFFERENTIAL CALCULUS-I)

TUTORIAL-I

SHORT QUESTIONS

1. Find the n^{th} derivative of $\frac{ax+b}{cx+d}$ Ans: $\frac{(-1)^n n!(bc-ad)c^{n-1}}{(cx+d)^{n+1}}$ (2013)
2. Find the 8^{th} derivative of the x^2e^{ax} . Ans: $e^x[x^2+16x+56]$ (2013)
3. Find the n^{th} derivative of $y = x^{n-1}logx$ Ans.: $\frac{(n-1)!}{x}$ (2012)
4. If $y = e^{asin^{-1}x}$, find the value of $(1-x^2)y_2 - xy_1 - a^2y$ Ans.: 0 (2015-16)
5. If $y = \cos^{-1} x$, then show that $(1-x^2)y_2 - xy_1 = 0$ (2022-23)

LONG QUESTIONS

6. Find the n^{th} derivative of $\frac{x^2}{(x+2)(2x+3)}$ Ans: $\frac{(-1)^n n!}{2}[\frac{9.2^n}{(2x+3)^{n+1}} - \frac{8}{(x+2)^{n+1}}]$. (2011)
7. If $\mu = sinnx + cosnx$, prove that $\mu_r = n^r[1+(-1)^r sin2nx]^{1/2}$, where μ_r denotes the r^{th} differential co-efficient of μ w.r.t. x. (2012) (2017-18)
8. If $y = [x+\sqrt{1+x^2}]^m$, find $y_n(0)$. (2013) (2021-22)
9. If $cos^{-1}x = log(y)^{\frac{1}{m}}$, then show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$ and hence calculate y_n when x=0. (2015-16)

TUTORIAL-II

LONG QUESTIONS

1. If $y = e^{mcos^{-1}x}$ show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$. And hence calculate y_n, when x=0. (2014) (2019-20)
2. If $y\sqrt{x^2-1} = \log_e(x+\sqrt{x^2-1})$, then show that $(x^2-1)y_{n+2} + (2n+1)xy_n + (n^2)y_{n-1} = 0$ (2022-23)
3. If $y = e^{\tan^{-1}x}$ show that $(x^2+1)y_{n+2} + [2(n+1)x-1]y_{n+1} + n(n+1)y_n = 0$. (2016-17)
4. If $y = \sin(m\sin^{-1}x)$ show that $(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$. And hence calculate y_n, when x=0 . (2018-19) (2020-21)

TUTORIAL-III

SHORT QUESTIONS

1. If $V = (x^2 + y^2 + z^2)^{-1/2}$, then find $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z}$ Ans.: -V (2015-16)
2. State Euler's Theorem on homogeneous function. (2021-22)
3. By using Euler's theorem to show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ for $u = \tan^{-1}(x^2 + 2y^2)$.

4. If $u = \sin^{-1}\left(\frac{x^3 + y^3}{\sqrt{x} + \sqrt{y}}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - \frac{5}{2}\tan u = 0$. (2022-23)

LONG QUESTIONS

5. If $u = sin^{-1}\left(\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{6}}+y^{\frac{1}{6}}}\right)$ then evaluate the value of $\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$ (2016-17)
6. If $I_n = \frac{d^n}{dx^n}(x^n logx)$, show that $I_n = nI_{n-1} + (n-1)!$ (2016-17)
7. If $\sin^{-1} y = 2\log(x+1)$ show that $(1-x)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2-m^2)\, y_n = 0$. And hence calculate y_n, when $x=0$. (2018-19)
8. If $x^x y^y z^z = c$ show that at $x=y=z$, $\frac{\partial^2 z}{\partial x\partial y} = -(x \log ex)^{-1}$. (2013-14)
9. If $u = f(r)$ where $r^2 = x^2 + y^2$ show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$. (2006, 2007, 2010, 2015-16)

TUTORIAL-IV

SHORT QUESTIONS

1. If $z=u^2+v^2$ and $u=at^2, v=2at$, then find $\frac{dz}{dt}$. (2014)
2. Find the condition for the contour on x-y plane where the partial derivative of (x^2+y^2) w.r.t. y is equal to the partial derivative of $(6y+4x)$ w.r.t. x. (2016-17)
3. Find $\frac{du}{dt}$ if $u = x^3 + y^3$ where $x = a\cos t,\ y = b\sin t$. (2019-20)
4. If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$. (2020-21)

LONG QUESTIONS

5. Verify Euler's theorem for (i) $u = \frac{x(x^4-y^4)}{x^4+y^4}$ (ii) $u = \frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$ (2014)
6. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$. (2018-19) (2019-20)
7. If $u = sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$. Also evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial xy} + y^2\frac{\partial^2 u}{\partial y^2}$. (2020-21)
8. $u = f(2x-3y, 3y-4z, 4z-2x)$ Prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$. (2019-20)

4/12/23 Unit - II (Differential Calculus - I)

Tutorial - I

Short

1. find nth derivative of $\frac{ax+b}{cx+d}$.

$cx+d \overline{)\,ax+b\,}(\frac{a}{c}$
$ax+\frac{ad}{c}$
$\frac{bc-ad}{c}$

Ans $y = \frac{a}{c} + \frac{bc-ad}{(cx+d)c}$

$y_n = 0 + \left(\frac{bc-ad}{c}\right) D^n\left(\frac{1}{cx+d}\right)$

$= \left(\frac{bc-ad}{c}\right) \frac{(-1)^n n!\, c^n}{(cx+d)^{n+1}} = \frac{(-1)^n n!\, c^{n-1}(bc-ad)}{(cx+d)^{n+1}}$

2. Find 8th derivative of $x^2 e^{ax}$

Ans $y = \underset{v}{x^2}\, \underset{u}{e^{ax}}$

$y_n = {}^nC_0 (D^n e^{ax})\, x^2 + {}^nC_1 (D^{n-1} e^{ax})\, Dx^2 + {}^nC_2 (D^{n-2} e^{ax})\, D^2 x^2$

Putting $n=8 \Rightarrow y_8 = {}^8C_0 (D^8 e^{ax})\, x^2 + {}^8C_1 (D^7 e^{ax})\, Dx^2 + {}^8C_2 (D^6 e^{ax})\, D^2 x^2$

$y_8 = {}^8C_0 a^8 e^{ax} . x^2 + 8a^7 e^{ax} . 2x + \frac{8\times 7}{2} \times 2\, e^{ax} . a^6$

$y_8 = a^8 e^{ax} . x^2 + 16 a^7 e^{ax} . x + 56 a^6 e^{ax} = e^{ax} . a^6 [a^2x^2 + 16ax + 56]$

3. find nth derivative of $y = x^{n-1} \log x$ —(1)

Ans Diff. w.r.t x

$y_1 = (n-1)x^{n-2} \log x + x^{n-1} \times \frac{1}{x} = (n-1)x^{n-2}\log x + x^{n-2} = x^{n-2}[1 + (n-1)\log x]$

$y_1 = \frac{x^{n-1}}{x} + \frac{y}{x}(n-1) \Rightarrow xy_1 = x^{n-1} + y(n-1)$ —(2)

Again diff. (2) using Leibnitz $(n-1)$ times

${}^{n-1}C_0 (D^{n-1} y_1)\, x + {}^{n-1}C_1 (D^{n-2} y_1)\, Dx = D^{n-1}(x^{n-1}) + (n-1) D^{n-1} y$

$y_n x + \cancel{(n-1) y_{n-1}} = (n-1)! + \cancel{(n-1) y_{n-1}}$

$y_n = \frac{(n-1)!}{x}$

4. If $y = e^{a \sin^{-1} x}$ —(1), find value of $(1-x^2)y_2 - xy_1 - a^2 y = ?$

Ans Diff. w.r.t x

$y_1 = e^{a\sin^{-1}x} \frac{a}{\sqrt{1-x^2}} \Rightarrow (\sqrt{1-x^2})\, y_1 = ay$

Squaring both sides $(1-x^2) y_1^2 = a^2 y^2$ —(2)

Again diff. w.r.t x $(1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = a^2\, 2y y_1 \Rightarrow (1-x^2) y_2 - xy_1 = a^2 y$

$(1-x^2) y_2 - xy_1 - a^2 y = 0$

5. If $y = \cos^{-1}x$ (1). then show that $(1-x^2)y_2 - xy_1 = 0$.

Ans Diff. w.r.t x $\quad y_1 = \frac{-1}{\sqrt{1-x^2}} \Rightarrow y_1(\sqrt{1-x^2}) = -1$

Squaring both sides $\quad y_1^2(1-x^2) - 1 = 0$ — (2).

Again diff. w.r.t x $\quad y_1^2(-2x) + (1-x^2)2y_1y_2 = 0$

$(1-x^2)y_2 - xy_1 = 0$

Long questions

6. Find nth derivative of $\frac{x^2}{(x+2)(2x+3)} = y$.

Ans To split y into partial fraction. Let $x+2 = z \Rightarrow 2x+3 = 2z-1$

$$y = \frac{(z-2)^2}{z(2z-1)} = \frac{z^2+4-4z}{z(2z-1)} = \frac{z}{(2z-1)} + \frac{4}{z(2z-1)} - \frac{4}{(2z-1)} = \frac{(2z-1)+1}{2(2z-1)} + \frac{4}{z(2z-1)} - \frac{4}{(2z-1)}$$

$$y = \frac{1}{2(2z-1)} + \frac{1}{2} + \frac{4}{z(2z-1)} - \frac{2}{(2z-1)} = \frac{5}{2(2z-1)} + \frac{1}{2} - \frac{2}{(2z-1)} = \frac{5}{2(2x+3)} + \frac{1}{2} - \frac{2}{(2x+3)}$$

$$y_n = \frac{5(-1)^n n!\, 2^n}{2(2x+3)^{n+1}} - \frac{2(-1)^n n!\, 2^n}{(2x+3)^{n+1}} = (-1)^n n!$$

$$y = \frac{1}{2} + \frac{4}{z(2z-1)} + \frac{1}{(2z-1)}\left(\frac{1}{2} - 2\right) = \frac{1}{2} + \frac{4}{z(2z-1)} - \frac{3}{(2z-1)2}$$

$$y = \frac{1}{2} + \frac{4z-2+2}{z^2(2z-1)} - \frac{3}{(2z-1)2} = \frac{1}{2} + \frac{2(2z-1)+2}{z^2(}$$

$$y = \frac{1}{2} + \frac{1}{2(2z-1)} + \frac{8}{(2z-1)} - \frac{8}{2z} - \frac{4}{(2z-1)} = \frac{1}{2} + \frac{1}{2(2z-1)} + \frac{4}{(2z-1)} - \frac{8}{2z}$$

$$y = \frac{1}{2} + \frac{9}{2(2z-1)} - \frac{8}{2z} = \frac{1}{2} + \frac{9}{2(2x+3)} - \frac{4}{(x+2)}$$

$$y_n = 0 + \frac{9}{2} D^n\left(\frac{1}{2x+3}\right) - \frac{4}{1} D^n\left(\frac{1}{x+2}\right) = \frac{9}{2}\left[\frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}}\right] - \frac{8}{2}\left[\frac{(-1)^n n!(1)^n}{(x+2)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!}{2}\left[\frac{9 \cdot 2^n}{(2x+3)^{n+1}} - \frac{8}{(x+2)^{n+1}}\right]$$

7. If $u = \sin nx + \cos nx$ prove. $u_r = n^r[1+(-1)^r \sin 2nx]^{1/2}$, u_r is rth differential co-efficient of u w.r.t x.

Ans Diff'g rth times, $u_r = D^r(\sin nx) + D^r(\cos nx) = n^r \sin\left(nx + \frac{r\pi}{2}\right) + n^r \cos\left(nx + \frac{r\pi}{2}\right)$

$$u_r = n^r\left[\sin\left(nx+\frac{r\pi}{2}\right) + \cos\left(nx + \frac{r\pi}{2}\right)\right] \quad (1)$$

$$u_r = n^r\left[\left(\sin\left(nx+\frac{r\pi}{2}\right) + \cos\left(nx+\frac{r\pi}{2}\right)\right)^2\right]^{1/2}$$

$$= n^r\left[1 + \sin 2\left(nx + \frac{r\pi}{2}\right)\right]^{1/2} = n^r\left[1 + \sin(2xn + r\pi)\right]^{1/2}$$

$$= n^r[1 + (-1)^r \sin 2nx]^{1/2} \quad \#$$

8. If $y = [x+\sqrt{1+x^2}]^m$ — ① find $y_n(0)$.

Ans Taking log both sides

$\log y = m \log[x+\sqrt{1+x^2}]$

diff^{ng} w.r.t x.

$\frac{1}{y} y_1 = m \times \frac{1}{(x+\sqrt{1+x^2})} \times \left(1+ \frac{1}{2\sqrt{1+x^2}} \times 2x\right) = m \times \frac{1}{\cancel{x+\sqrt{1+x^2}}} \left(\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}\right)$

$\frac{1}{y} y_1 = \frac{m}{\sqrt{1+x^2}} \Rightarrow y_1 = \frac{my}{\sqrt{1+x^2}}$ — ② $\Rightarrow y_1\sqrt{1+x^2} = my$

Squaring both sides

$(1+x^2)y_1^2 = m^2y^2 \Rightarrow (1+x^2)\, 2y_2 y_1 + y_1^2 2x = m^2 2y y_1$

$\Rightarrow \cancel{2y_1}[(1+x^2)y_2 + xy_1] = \cancel{2y_1}(m^2y) \Rightarrow (1+x^2)y_2 + xy_1 - m^2y = 0$ — ③

Apply Leibnitz theorem

$[{}^nC_0 y_{n+2}(1+x^2) + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n (2)] + [{}^nC_0 y_{n+1} x + {}^nC_1 y_n \times 1] - m^2 y_n = 0$

$\boxed{(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0}$ — ④

Put $x=0$, in ①, ②, ③, ④ $\quad y(0) = 1 \quad , y_1(0) = m(y_0), y_2(0) = m^2 y(0)$

$y_1(0) = m \quad , y_2(0) = m^2$

Put $n = 1, 2, 3, 4 \ldots$

$n=1 \quad y_3(0) = (m^2-n^2)\, y_1(0) = -(1-m^2)\, m$

$n=2 \quad y_4(0) = -(2^2-m^2)y_2(0) = -(2^2-m^2)m^2$

$n=3 \quad y_5(0) = -(3^2-m^2)\, y_3(0) = +(3^2-m^2)(1^2-m^2)m$

$y_6(0) = -(4^2-m^2)y_4(0) = (4^2-m^2)(2^2-m^2)m^2$

$$y_n = \begin{cases} [(n-2)^2-m^2][(n-4)^2-m^2] \ldots\ldots (3^2-m^2)(1^2-m^2)\, m \text{ ; n is odd} \\ [(n-2)^2-m^2][(n-4)^2-m^2] \ldots\ldots (4^2-m^2)(2^2-m^2)\, m^2\, (-1)^{n-1} \text{ when n is even} \end{cases}$$

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9. If $\cos^{-1}x = \log(y)^{1/m}$ then show that $(1-x^2)y_{n+2} - (2n+1)y_{n+1}x - (n^2+m^2)y_n = 0$.

& hence calculate y_n when $x=0$.

Ans $\cos^{-1}x = \frac{1}{m}\log y \Rightarrow m\cos^{-1}x = \log y \Rightarrow y = e^{m\cos^{-1}x}$ — ①

Diff. w.r.t x we get

$y_1 = e^{m\cos^{-1}x}\left(\frac{-m}{\sqrt{1-x^2}}\right) \Rightarrow (\sqrt{1-x^2})y_1 = -me^{m\cos^{-1}x} \Rightarrow -my = \sqrt{1-x^2}y_1$ — ②

Squaring both sides $\quad (1-x^2)y_1^2 = m^2y^2$

Again Diff w.r.t x $\quad (-2x)y_1^2 + (1-x^2)2y_2y_1 = m^2 2y\, y_1 x$

$-xy_1 + (1-x^2)y_2 = m^2 y$ — (3)

Using Leibnitz theorem $D^n[(1-x^2)y_2] - D^n(xy_1) - m^2 D^n[y] = 0$

$$\left[{}^nC_0(1-x^2)[D^n y_2] + {}^nC_1[D(1-x^2)(D^{n-1}y_2)] + {}^nC_2[D^2(1-x^2)(D^{n-2}y_2)]\right] - \left[{}^nC_0(D^n y_1)x + {}^nC_1 D^{n-1}y_1 Dx\right] - m^2 y_n = 0$$

$$[y_{n+2}(1-x^2) + ny_{n+1}(-2x) + n(n-1)y_n] - [y_{n+1}x + ny_n] - m^2 y_n = 0$$

$$\boxed{y_{n+2}(1-x^2) - y_{n+1}(2n+1)x - (n^2+m^2)y_n = 0}$$ proved (4)

Put $x=0$ Eq (1), (2), (3), (4)

$y(0) = e^{m\pi/2}$, $y_1(0) = -me^{m\pi/2}$, $y_2(0) = m^2 e^{m\pi/2}$

Put $n = 1, 2, 3, 4$ in eqn (4)

$n=1$ $y_3(0) = (1^2+m^2)(-m).e^{m\pi/2}$

$n=2$ $y_4(0) = (2^2+m^2)\, m^2 e^{m\pi/2}$

$n=3$ $y_5(0) = (3^2+m^2)(1^2+m^2) - m.e^{m\pi/2}$

$n=4$ $y_6(0) = (4^2+m^2)(2^2+m^2)\, m^2 e^{m\pi/2}$, ... so on

In general
$$y_n(0) = \begin{cases} -me^{m\pi/2}(1^2+m^2)(3^2+m^2)\cdots\cdots[(n-2)^2+m^2], & n \text{ is odd} \\ m^2 e^{m\pi/2}(2^2+m^2)(4^2+m^2)\cdots\cdots((n-2)^2+m^2), & n \text{ is even} \end{cases}$$

Long questions

1. If $y = e^{m\cos^{-1}x}$ show that $(1-x^2)y_{n+2} - y_{n+1}x(2n+1) - (n^2+m^2)y_n = 0$ & find $y_n(0) = ?$

Ans Refer above solution

2. If $y\sqrt{x^2-1} = \log_e(x+\sqrt{x^2-1})$ — (1) then show that $(x^2-1)y_{n+1} + (2n+1)xy_n + (n^2)y_{n-1} = 0$

Ans Squaring both sides $y^2(x^2-1) = [\log_e(x+\sqrt{x^2-1})]^2$

Diffng w.r.t x $(\cancel{2}x)y^2 + (x^2-1)\cancel{2}yy_1 = \cancel{2}\log_e(x+\sqrt{x^2-1}) \cdot \frac{1}{(x+\sqrt{x^2-1})} \cdot \left(1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right)$

$\Rightarrow$ $xy^2 + (x^2-1)\cancel{y}y_1 = \frac{\log(x+\sqrt{x^2-1})}{\sqrt{x^2-1}} = \cancel{y}$

$\Rightarrow$ $xy + y_1(x^2-1) - 1 = 0$ — (2)

Again diff. w.r.t x $y + y_1(2x) + (x^2-1)y_2 - 0 = 0$

$y + y_1(2x) + y_2(x^2-1) = 0$ — (3)

Using Leibnitz in eqⁿ ②

$$D^n(xy) + D^n y_1(x^2-1) - D^n(1) = 0$$

$$\left[{}^nC_0 D^n y \cdot x + {}^nC_1 D^{n-1} y\, Dx\right] + \left[{}^nC_0 D^n y_1 (x^2-1) + {}^nC_1 D^{n-1} y_1\, D(x^2-1) + {}^nC_2 D^{n-2} y_1\, D^2(x^2-1)\right] = 0$$

$$y_n x + y_{n-1} n + y_{n+1}(x^2-1) + y_n n(2x) + \frac{n(n-1)}{2!} \cdot y_{n-1} \cdot 2 = 0$$

$$y_n x + \cancel{y_{n-1} n} + y_{n+1}(x^2-1) + y_n n(2x) + n^2 y_{n-1} - \cancel{n y_{n-1}} = 0$$

$$(x^2-1) y_{n+1} + x y_n (2n+1) + n^2 y_{n-1} = 0$$

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3. If $y = e^{\tan^{-1}x}$ —① . show that $(x^2+1)y_{n+2} + [2(n+1)x - 1]\, y_{n+1} + n(n+1) y_n = 0$.

Ans Diff. w.r.t x: $y_1 = e^{\tan^{-1}x}\left(\frac{1}{1+x^2}\right) \Rightarrow (1+x^2) y_1 = e^{\tan^{-1}x} = y$ —②

Again diff w.r.t x: $y_2(1+x^2) + y_1(2x) = y_1 \Rightarrow y_2(x^2+1) + y_1(2x-1) = 0$ —③

Using Leibnitz: $D^n[y_2(x^2+1)] + D^n[y_1(2x-1)] = 0$

or, $\left[{}^nC_0 D^n y_2 (x^2+1) + {}^nC_1 D^{n-1} y_2\, D(x^2+1) + {}^nC_2 D^{n-2} y_2\, D^2(x^2+1)\right] + \left[{}^nC_0 D^n y_1 (2x-1) + {}^nC_1 D^{n-1} y_1\, D(2x-1)\right] = 0$

or, $y_{n+2}(x^2+1) + n y_{n+1}(2x) + \frac{n(n-1)}{2!} y_n (2) + y_{n+1}(2x-1) + n y_n (2) = 0$

or, $y_{n+2}(x^2+1) + y_{n+1}(2nx) + n^2 y_n - n y_n + y_{n+1}(2x) - y_{n+1} + 2n y_n = 0$

$$y_{n+2}(x^2+1) + y_{n+1}[2(n+1)x - 1] + n(n+1) y_n = 0$$

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4. If $y = \sin(m \sin^{-1} x)$ —① show $(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$. & find $y_n(0)$.

Ans Diff. w.r.t x: $y_1 = \cos(m\sin^{-1}x) \cdot \frac{m}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\, y_1 = \cos(m\sin^{-1}x)\, m$

Squaring both sides: $(1-x^2) y_1^2 = \cos^2(m\sin^{-1}x) \cdot m^2$

$(1-x^2) y_1^2 = m^2[1 - \sin^2(m\sin^{-1}x)] = m^2(1-y^2)$ —②

Again diff w.r.t x: $(1-x^2)\cancel{2} y_1 y_2 + y_1^2(-\cancel{2}x) = m^2(-\cancel{2} y y_1)$

$(1-x^2) y_2 - y_1 x = -m^2 y$

$(1-x^2) y_2 - y_1 x + m^2 y = 0$

Using Leibnitz theorem

$$\left[y_{n+2}(1-x^2)+y_{n+1}(-2x)n+\frac{n(n-1)}{2!}y_n(-2)\right]-\left[y_{n+1}x+ny_n\right]+m^2y_n=0$$

$$\boxed{y_{n+2}(1-x^2)-y_{n+1}x(2n+1)-(n^2-m^2)y_n=0}$$

$y=0,\ y_1=m,$
$y_2=0,\ y_3=m(1^2-m^2)$
$y_4=0,\ y_5=(9^2-m^2)(1-m^2)m$

$$y_n(0)=\begin{cases}0 & \text{even } n\\ m(1^2-m^2)(3^2-m^2)(5^2-m^2)\ldots((n-2)^2-m^2) & \text{odd}\end{cases}$$

Tutorial - III

1. $V=(x^2+y^2+z^2)^{-1/2}$. then find $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}$?

Ans $\frac{\partial V}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.(2x+0+0)=-x(x^2+y^2+z^2)^{-3/2}=-x(V^{-2})^{-3/2}$

$\frac{\partial V}{\partial y}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(0+2y+0)=-y(x^2+y^2+z^2)^{-3/2}=-y(V^{-2})^{-3/2}$

$\frac{\partial V}{\partial z}=-z(V^{-2})^{-3/2}$, similarly

$x\frac{\partial V}{\partial x}=-x^2(V^{+3/2})$, $y\frac{\partial V}{\partial y}=-y^2V^{+3/2}$, $z\frac{\partial V}{\partial z}=-z^2V^{+3/2}$

$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=-V^{3/2}(x^2+y^2+z^2)=V^3(V^{-2})=-V$

2. State Euler theorem on homogeneous function.
A function is said to be homogeneous of degree n in x & y. If it can be expressed as $x^n f(y/x)$.

3. By using euler theorem show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$ for $u=\tan^{-1}(x^2+2y^2)$.

Ans. $u=\tan^{-1}(x^2+2y^2)$ — ①

Replacing x by tx & y by ty $\quad u=\tan^{-1}t^2(x^2+2y^2)$ not a homogeneous function

Now, $f(u)=\tan u=(x^2+2y^2)\Rightarrow$ this is homogeneous function of degree $n=2$

By euler theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{nf(u)}{f'(u)}=\frac{2\tan u}{\sec^2 u}=\sin 2u$

4. If $u=\sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}-\frac{5}{2}\tan u=0$.

Ans u is not homogeneous function here, but $\sin u=\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}=f(x,y)$

Replacing x by tx & y by ty $\quad f(tx,ty)=\frac{t^3x^3+t^3y^3}{\sqrt{tx}+\sqrt{ty}}=t^{3-1/2}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)=t^{5/2}f(x,y)$

$\sin u$ is a homogeneous function of degree $n=5/2$

By euler theorem

$$x\frac{\partial(\sin u)}{\partial x}+y\frac{\partial \sin u}{\partial y} = \frac{5}{2}\sin u = \frac{+5}{2}\sin u$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y} = \frac{5}{2}\sin u \Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{5}{2}\tan u \Rightarrow 0 \text{ // } \#$$

Long questions

5. If $u = \sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$ then evaluate the value of $\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right)$

Ans u is not a homogeneous function. but,

$$\sin u = \frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}} = f(x,y) = f(u)$$

$$f(tx,ty) = \frac{t^{1/4}(x^{1/4}+y^{1/4})}{t^{1/6}(x^{1/6}+y^{1/6})} = t^{1/4-1/6}f(x,y) = t^{1/12}f(x,y)$$

$\therefore$ $\sin u$ is a homogeneous function of degree $n = 1/12$

By deduction from euler theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{nF(u)}{F'(u)} = \frac{1}{12}\cdot\frac{\sin u}{\cos u} = \frac{1}{12}\tan u = \phi(u)$$

$$\& \quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u)-1] = \frac{1}{12}\tan u\left[\frac{1}{12}\sec^2 u-1\right]$$

---

6. If $I_n = \frac{d^n}{dx^n}(x^n\log x)$ show that $I_n = nI_{n-1}+(n-1)!$

Ans

$$\cancel{I_n = {}^nC_0 D^n x^n \log x + {}^nC_1 D^{n-1}x^n D\log x + {}^nC_2 D^{n-2}x^n D^2\log x + \dots {}^nC_n x^n D^n \log x}$$

$$\cancel{= n!\log x + n(n-1)!\cdot\frac{1}{x} + \frac{n(n-1)}{2!}(n-2)!\cdot\left(\frac{-1}{x^2}\right)}$$

$$I_n = \frac{d^n}{dx^n}(x^n\log x) = \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}(x^n\log x)\right\}$$

Diff^{ng} w.r.t x

$$\frac{d^{n-1}}{dx^{n-1}}\left\{x^n\frac{1}{x}+\log x\cdot nx^{n-1}\right\} = \frac{d^{n-1}}{dx^{n-1}}x^{n-1}+n\frac{d^{n-1}}{dx^{n-1}}(\log x\cdot x^{n-1})$$

$$\boxed{I_n = (n-1)!+nI_{n-1}} \quad \#$$

---

7. If $\sin^{-1}y = 2\log(x+1)$. Show that $(1-x)y_{n+2}-(2n+1)y_{n+1}x-(n^2-n^2)y_n = 0$.
& find y_n at $x=0$.

Ans $y = \sin[2\log(x+1)]$ — ①

Diff w.r.t x in ① $\quad y_1 = \cos[2\log(x+1)]\times\left(\frac{2}{x+1}\right)$ — ②

$$(x+1)y_1 = 2\cos\left(2\log(x+1)\right) \quad \text{--- (2)}$$

<u>Again diff w.r.t x</u> $$(x+1)y_2 + y_1 = -2\sin(2\log(x+1)) \cdot \frac{2}{x+1}$$

$$(x+1)^2 y_2 + (x+1)y_1 + 4y = 0 \quad \text{--- (3)}$$

<u>Applying Leibnitz theorem</u>

$$\left[{}^nC_0 D^n(y_2) D^0(x+1)^2 + {}^nC_1 D^{n-1}(y_2) D^1(x+1)^2 + {}^nC_2 D^{n-2}(y_2) D^2(x+1)^2\right] + \left[{}^nC_0 D^n(y_1) D^0(x+1) + {}^nC_1 D^{n-1}(y_1) D^1(x+1)\right] + 4D^n y = 0$$

$$\left[y_{n+2}(x+1)^2 + n\, y_{n+1}\, 2(x+1) + \frac{n(n-1)}{2} y_n \cdot 2\right] + \left[y_{n+1}(x+1) + n\, y_n (1)\right] + 4y_n = 0$$

$$\boxed{y_{n+2}(x+1)^2 + y_{n+1}(2n+1)(x+1) + (n^2+4)y_n = 0} \quad \text{--- (4)}$$

<u>Put $x=0$ in eqn (1), (2), (3) & (4) we get</u>

$y(0) = 0$, $y_1(0) = 2$, $y_2(0) = -2$,

<u>Put $n = 1, 2, 3 \ldots$</u> in eqn (4),

$n=1$, $y_3 = -y_2(3) - 5y_1 = 6 - 10 = -4 = 1 \times (-1)^2$

$n=2$ $y_4 = -y_3(5) - 8y_2 = -(-4)5 - 8(-2) = 20 + 16 = 36$

$n=3$ $y_5 = -y_4(7) - 13y_3 = -36\times 7 - 13(-4) = -36\times 7 + 13\times 4 = 4(-63+13) = -50\times 4 = -200$

$n=4$ $y_6 = -y_5(9) - 20y_4 = -(-200)9 - 20\times 36 = 180(10-4) = 180\times 6 = 480$

$$y_n(0) = \begin{cases} 0, -2, 36, 480 \ldots, & n \text{ odd} \\ 2, -4, -200 \ldots, & n \text{ even} \end{cases}$$

8) If $x^x y^y z^z = c$ show that at $x=y=z$ $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

Ans taking log both sides $\log(x^x y^y z^z) = \log c$

$x \log x + y \log y + z \log z = 0$

Diff^{ng} w.r.t x (Assuming x & z as variable & y as constant)

$\log x + x \cdot \frac{1}{x} + 0 + z \cdot \frac{1}{z}\frac{\partial z}{\partial x} + \log z \frac{\partial z}{\partial x} = 0$

$(1+\log x) + \frac{\partial z}{\partial x}(1+\log z) = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(-\frac{(1+\log x)}{1+\log z}\right)$ [diff^{ng} $\frac{\partial z}{\partial x}$ w.r.t y]

$= -\left[\frac{0-(1+\log x)\left(0+\frac{1}{z}\frac{\partial z}{\partial y}\right)}{(1+\log z)^2}\right]$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log x)}{z(1+\log z)^2}\frac{\partial z}{\partial y} = \frac{(1+\log x)}{z(1+\log z)^2} \times \frac{-(1+\log y)}{(1+\log z)}$

At $x=y=z$ $= -\frac{(1+\log x)(1+\log x)}{x(1+\log x)^3} = \frac{-1}{x(1+\log x)} = \frac{-1}{x(\log e+\log x)}$

$= \frac{-1}{x \log ex} = -[x \log ex]^{-1}$

---

9) if $u = f(r)$ — (i) where $r^2 = x^2+y^2$ — (ii) show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

Ans $\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}$ (diff u w.r.t x) — (iii)

diff^{ng} eq^n (ii) w.r.t x $2r\frac{\partial r}{\partial x} = 2x \Rightarrow r\frac{\partial r}{\partial x} = x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

Same as diff^{ng} eq^n (ii) w.r.t y $\frac{\partial r}{\partial y} = \frac{y}{r}$

Put value of $\frac{\partial r}{\partial x}$ in eq^n (iii) $\frac{\partial u}{\partial x} = f'(r)\frac{x}{r}$

Again diff^{ng} w.r.t x $\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)(1)}{r} + x\left[\frac{rf''(r)\frac{\partial r}{\partial x} - f'(r)\frac{\partial r}{\partial x}}{r^2}\right]$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + \frac{x^2}{r^3}\left[rf''(r) - f'(r)\right] \quad \text{--- (4)}$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + \frac{y^2}{r^3}\left[rf''(r) - f'(r)\right] \quad \text{--- (5)}$$

Adding (4) & (5)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2\frac{f'(r)}{r} + \frac{\left[rf''(r) - f'(r)\right]}{r}\left(\frac{x^2+y^2}{r^2}\right) \quad \text{from eq}^n \text{ (1)}$$

$$= 2\frac{f'(r)}{r} + f''(r) - \frac{f'(r)}{r} = f''(r) + \frac{f'(r)}{r}$$

Tutorial-IV

Short questions

1. If $z = u^2+v^2$ & $u = at^2$, $v = 2at$ then find $\frac{dz}{dt}$.

Ans $$\frac{dz}{dt} = \frac{\partial z}{\partial u}\left(\frac{du}{dt}\right) + \frac{\partial z}{\partial v}\left(\frac{dv}{dt}\right) = \frac{\partial z}{\partial u}(2at) + \frac{\partial z}{\partial v}(2a) = (2u)(2at) + 2v(2a)$$

$$= 4a(ut+v)$$

---

3. find $\frac{du}{dt}$ if $u = x^3+y^3$ where $x = a\cos t$, $y = b\sin t$.

Ans $$\frac{du}{dt} = \frac{\partial u}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right) = \frac{\partial u}{\partial x}(a\sin t) + \frac{\partial u}{\partial y}(b\cos t) = 3x^2(-a\sin t) + 3y^2(b\cos t)$$

$$= 3(y^2 b\cos t - x^2 a\sin t)$$

---

4. If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

where $(X = y-z,\ Y = z-x,\ Z = x-y)$.

Ans Here, $u = f(X, Y, Z)$

$\therefore$ u is a composite function of x, y & z

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}\cdot(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1) \quad \text{--- (11)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0) \quad \text{--- (111)}$$

Adding (1), (11), (111) we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Imp question

5. Verify Euler theorem for (i) $u = \frac{x(x^4-y^4)}{x^4+y^4}$ (ii) $u = \frac{x^{1/3}+y^{1/3}}{y^{1/2}+x^{1/2}}$

Ans (i) $u(x,y) = \frac{x(x^4-y^4)}{x^4+y^4}$ — ① $u(tx,ty) = \frac{tx(t^4x^4-t^4y^4)}{(t^4x^4+t^4y^4)} = t\,u(x,y)$

u is a homogeneous function of degree n = 1 in x & y

∴ By Euler theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (1)\,u(x,y)$

from ①, $\frac{\partial u}{\partial x} = \frac{(x^4+y^4)[(x^4-y^4)+x\,4x^3-0] - x(x^4-y^4)(4x^3+0)}{(x^4+y^4)^2}$

$= \frac{(x^4+y^4)(5x^4-y^4) - (x^4-y^4)4x^4}{(x^4+y^4)^2}$

$= \frac{5x^8+5x^4y^4-x^4y^4-y^8-4x^8+4x^4y^4}{(x^4+y^4)^2} = \frac{x^8+8x^4y^4-y^8}{(x^4+y^4)^2}$

$x\frac{\partial u}{\partial x} = \frac{x^9+8x^5y^4-y^8x}{(x^4+y^4)^2}$

$\frac{\partial u}{\partial y} = \frac{(x^4+y^4)[0+x(0-4y^3)] - x(x^4-y^4)[0+4y^3]}{(x^4+y^4)^2}$

$= \frac{(x^4+y^4)(-4xy^3) - (x^4-y^4)4xy^3}{(x^4+y^4)^2} = \frac{-4x^5y^3-\cancel{4xy^7}-4x^5y^3+\cancel{4y^7x}}{(x^4+y^4)^2}$

$= -\frac{8x^5y^3}{(x^4+y^4)^2}$

$y\frac{\partial u}{\partial y} = -\frac{8x^5y^4}{(x^4+y^4)^2}$

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{x^9-xy^8}{(x^4+y^4)^2} = \frac{x(x^8-y^8)}{(x^4+y^4)^2} = \frac{x(x^4-y^4)\cancel{(x^4+y^4)}}{(x^4+y^4)^{\cancel{2}}} = u$

Hence verified.

(ii) $\frac{\partial u}{\partial x} = \frac{(x^{1/2}+y^{1/2})(\frac{1}{3}x^{-2/3}) - (x^{1/3}+y^{1/3})(\frac{1}{2}x^{-1/2})}{(x^{1/2}+y^{1/2})^2}$

$x\frac{\partial u}{\partial x} = \frac{\frac{1}{3}x^{1/3}(x^{1/2}+y^{1/2}) - (x^{1/3}+y^{1/3})\frac{1}{2}x^{1/2}}{(x^{1/2}+y^{1/2})^2} = \frac{\frac{1}{3}x^{5/6}+\frac{1}{3}x^{1/3}y^{1/2}-\frac{1}{2}x^{5/6}-\frac{1}{2}x^{1/2}y^{1/3}}{(x^{1/2}+y^{1/2})^2}$

$\frac{\partial u}{\partial y} = \frac{(x^{1/2}+y^{1/2})(\frac{1}{3}y^{-2/3}) - (x^{1/3}+y^{1/3})(\frac{1}{2}y^{-1/2})}{(x^{1/2}+y^{1/2})^2}$

$$y\frac{\partial u}{\partial y} = \frac{\frac{1}{3}y^{1/3}(x^{1/2}+y^{1/2}) - (x^{1/3}+y^{1/3})(\frac{1}{2}y^{1/2})}{(x^{1/2}+y^{1/2})^2} = \frac{\frac{1}{3}y^{1/3}x^{1/2} + \frac{1}{3}y^{5/6} - \frac{1}{2}x^{1/3}y^{1/2} - \frac{1}{2}y^{5/6}}{(x^{1/2}+y^{1/2})^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-\frac{1}{6}\left(x^{5/6} + y^{5/6} + x^{1/3}y^{1/2} + x^{1/2}y^{1/3}\right)}{(x^{1/2}+y^{1/2})^2}$$

$$= -\frac{1}{6}\,\frac{(x^{1/3}x^{1/2} + y^{1/3}y^{1/2} + x^{1/3}y^{1/2} + x^{1/2}y^{1/3})}{(x^{1/2}+y^{1/2})^2}$$

$$= -\frac{1}{6}\,\frac{x^{1/3}(x^{1/2}+y^{1/2}) + y^{1/3}(x^{1/2}+y^{1/2})}{(x^{1/2}+y^{1/2})^2} = -\frac{1}{6}u$$

6. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

Ans u is not homogeneous function, $\therefore f(x,y) = \cos u = (x+y/\sqrt{x}+\sqrt{y})$

$f(x,y)$ is homogeneous of degree $n = \frac{1}{2}$

$\therefore$ By euler theorem, $x\frac{\partial f(x,y)}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}f(x,y)$

$$x\frac{\partial \cos u}{\partial x} + y\frac{\partial \cos u}{\partial y} = \frac{1}{2}\cos u \Rightarrow -x\sin u\frac{\partial u}{\partial x} - \cos u\, y\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

7. If $u = \sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$, find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = ?$

Ans u is not homogeneous function, let $V(x,y) = \sec u = (x^3+y^3/x+y)$

$V(x,y)$ is homogeneous function of degree $n = 2$.

$\therefore$ By euler theorem $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 2V \Rightarrow x\frac{\partial \sec u}{\partial x} + y\frac{\partial \sec u}{\partial y} = 2\sec u$

$$x\sec u\tan u\frac{\partial u}{\partial x} + y\sec u\tan u\frac{\partial u}{\partial y} = 2\sec u \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$$

So, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cot u(-2\operatorname{cosec}^2 u - 1)$

8. $u = f(2x-3y, 3y-4z, 4z-2x)$ prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.

Ans let $X = 2x-3y$, $Y = 3y-4z$, $Z = 4z-2x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\left(\frac{\partial X}{\partial x}\right) + \frac{\partial u}{\partial Y}\left(\frac{\partial Y}{\partial x}\right) + \frac{\partial u}{\partial Z}\left(\frac{\partial Z}{\partial x}\right) = \frac{\partial u}{\partial X}(2) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-2)$$

$$\frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\left(\frac{\partial X}{\partial y}\right) + \frac{\partial u}{\partial Y}\left(\frac{\partial Y}{\partial y}\right) + \frac{\partial u}{\partial Z}\left(\frac{\partial Z}{\partial y}\right) = \frac{\partial u}{\partial X}(-3) + \frac{\partial u}{\partial Y}(3) + \frac{\partial u}{\partial Z}(0)$$

$$\frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\left(\frac{\partial X}{\partial z}\right) + \frac{\partial u}{\partial Y}\left(\frac{\partial Y}{\partial z}\right) + \frac{\partial u}{\partial Z}\left(\frac{\partial Z}{\partial z}\right) = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-4) + \frac{\partial u}{\partial Z}(4)$$

$$\frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} \quad — (3)$$

Adding (1), (2), (3) $\quad \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$

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2. Find the condition for the contour on x-y plane where the partial derivative of (x^2+y^2) w.r.t y is equal to the partial derivative of $(6y+4x)$ w.r.t x.

Ans $\quad \frac{\partial(x^2+y^2)}{\partial y} = \frac{\partial(6y+4x)}{\partial x} \Rightarrow 2y = 4 \Rightarrow \boxed{y=2}$